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Mh4714 Week 9

Week 9

0.1 Differentiation (Contd.)

Theorem 0.1 (Rolle's Theorem)

Let f be continuous over $[a, b]$ and differentiable over (a, b). If $f(a) = f(b)$ then there is a point $c \in (a, b)$ with $f'(c) = 0$.

Proof

f continuous over $[a, b] \Rightarrow f$ has a maximum value and a minimum value in $[a, b]$.

The possibilities are:

- The maximum and/or minimum value occurs in $(a, b) \Rightarrow$ there is $c \in (a, b)$ with $f'(c) = 0$. (This is a theorem that we proved above.)
- The maximum and minimum values both occur at an end point.
  But since $f(a) = f(b)$ then the maximum and minimum values are both the same in this case. This means that f is constant over $(a, b) \Rightarrow f'(x) = 0, \forall x \in (a, b)$.

□

0.1.0.1 Applications of Rolle's Theorem:.

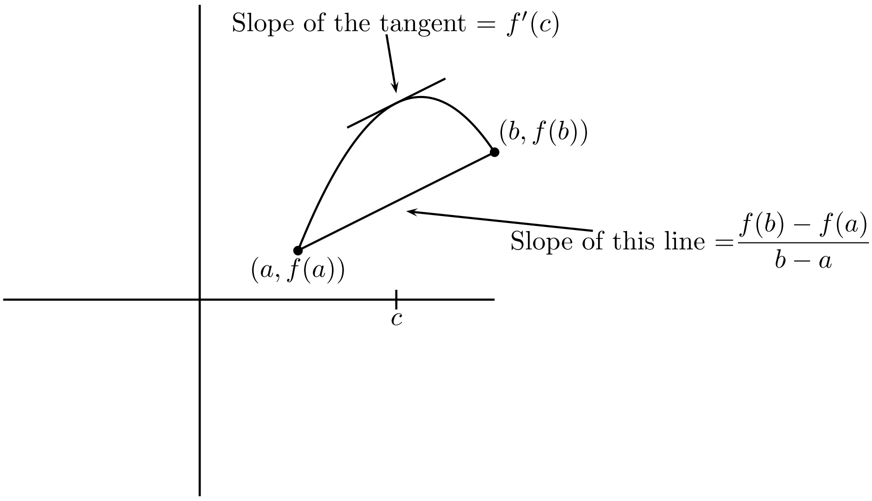
Let $f(x) = x^5 + x^3 - 3$. We can use the IMVT to prove that f has a root in the inverval $[1, 2]$ because $f(1) = -1, f(2) = 37$. We can use Rolle's theorem to prove that f does not have a second root in $[1, 2]$ because, if there were two roots, say, c_1 and c_2 then we would have $f(c_1) = f(c_2)$ and then Rolle's theorem would imply that there was some point c between c_1 and c_2 with $f'(c) = 0$. But $f'(x) = 5x^4 + 3x^2$ which is always positive over (1,2).

The important theorem known as the *Mean Value Theorem* is proved using Rolle's Theorem.

Theorem 0.2 (Mean Value Theorem)

Let f be continuous over $[a, b]$ and differentiable over (a, b). There is a point $c \in (a, b)$ with:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Proof

The equation of the line joining the points $(a, f(a))$ and $(b, f(b))$ is

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

i.e.

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

Let

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Now g is continuous over $[a, b]$ and differentiable over (a, b) with:

$$g(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0$$

and

$$g(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = f(b) - f(a) - (f(b) - f(a)) = 0$$

That is, g(a)=g(b).
Therefore by Rolle's Theorem there is $c \in (a, b)$ with $g'(c) = 0$.
And since $g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$, this gives $g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0$
That is, $f'(c) = \frac{f(b) - f(a)}{b - a}$.

□

0.1.0.2 Application of the Mean Value Theorem:.

It is easy to show that if a function f is constant over an interval (a, b) then $f'(x) = 0, \ \ \forall x \in (a, b)$.
We can now show that the converse of this is also true:

Theorem 0.3

If $f'(x) = 0, \ \ \forall x \in (a, b)$ then f is constant over (a, b).

Proof

Pick any two points x_1 and x_2 in the interval (a, b) and we can show that $f(x_1) = f(x_2)$ as follows:
Taking it that $x_1 < x_2$ we have that f is continuous over $[x_1, x_2]$ and differentiable over (x_1, x_2) and so, by the Mean Value Theorem there is a point $c \in ((x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

But $f'(c) = 0$ because $f'(x) = 0 \ \ \forall x \in (a, b)$ and so

$$0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \Rightarrow f(x_1) = f(x_2).$$

Therefore, f is constant over (a, b). □

Corollary 0.4

If $f'(x) = g'(x)$ for all $x \in (a, b)$ then there is some constant c with $f(x) = g(x) + c$.

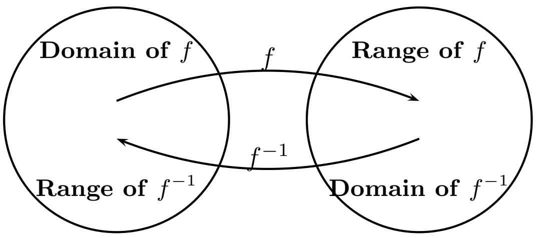
0.1.1 Inverse Functions

Some functions can be inverted unambiguously. That is, given y we can find x such that $f(x) = y$.

Example 0.5

Let $f(x) = 3x - 2$.
Then $y = 3x - 2 \Rightarrow x = \frac{1}{3}y + \frac{2}{3}$.
This gives us another function $g(y) = \frac{1}{3}y + \frac{2}{3}$.
g and f are said to be inverses of one another and have the defining feature that $f(g(y)) = y$ and $g(f(x)) = x$. g is frequently denoted as f^{-1}.

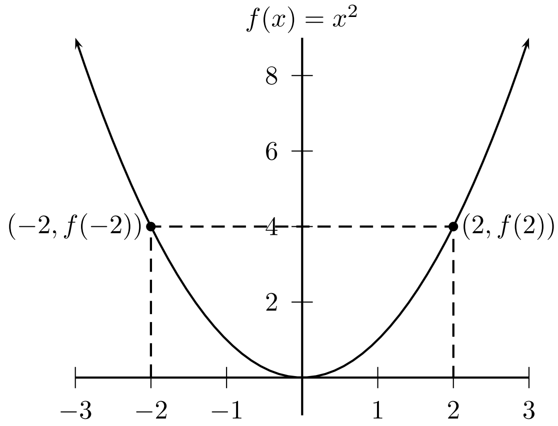
The following arrow diagram illustrates the relationships between the domain and range of f and the domain and range of f^{-1}



Not every function has an inverse.

Example 0.6

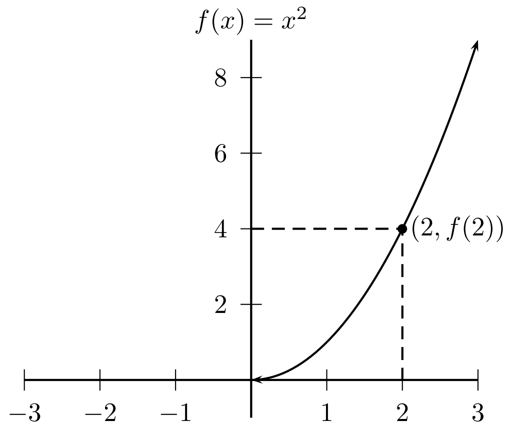
Let $f(x) = x^2$.
Then $y = x^2 \Rightarrow x = \pm\sqrt{y}$.
And, except in the case $y = 0$, there are two x's matched with each y and we do not have an inverse functon.



However, a function defined by the same formula but a different domain may have an inverse.

Example 0.7

Let $f(x) = x^2, x \in [0, \infty)$
then $y = x^2 \Rightarrow x = \sqrt{y}$ because only positive x's are in the domain of this function.



It is clear that a funtion has an inverse if and only if it is *one-to-one* or *injective*, that is, each x is matched with a distinct y.

Example 0.8

If n is an odd positive integer, the function x^n is injective and has inverse $x^{\frac{1}{n}}$
If n is an even positive integer, the function $x^n, x \in [0, \infty)$ is injective and has inverse $x^{\frac{1}{n}}, x \in [0, \infty)$

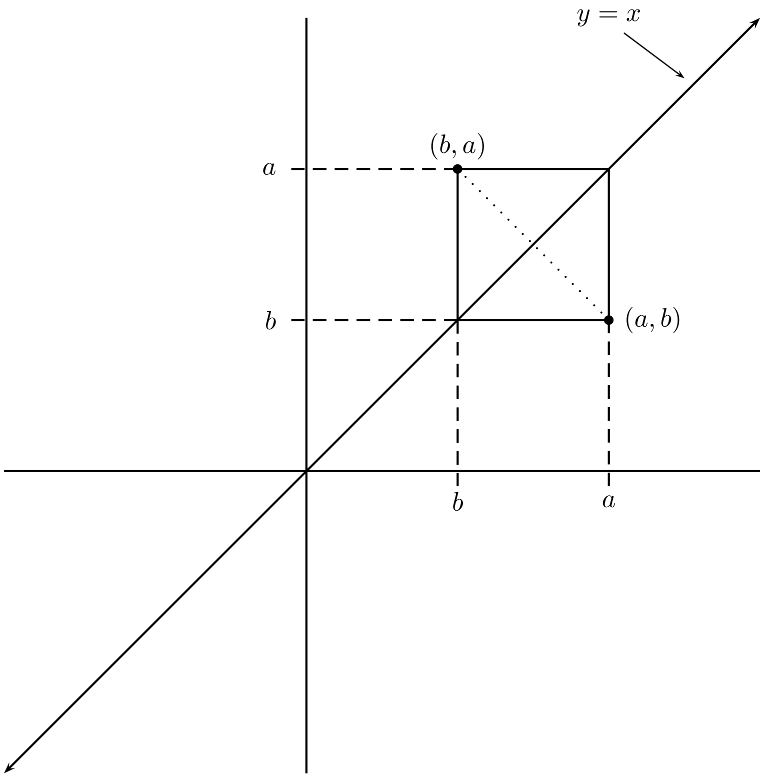
Properties of inverse functions:
Let f be injective with inverse f^{-1}.

(i) $f(f^{-1})(x)$ for all x in the domain of f^{-1} and
$f^{-1}(f(x)) = x$ for all x in the domain of f.

(ii) If f is continuous at $a \in \mathbb{R}$ then f^{-1} is continuous at $f(a)$.

(iii) The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.

Proof



Beginning with any point (a, b) not on the line $y = x$, construct a square, as above, one of whose diagonals is part of the $y = x$ line.
It is then clear that the other diagonal of the square joins the points (a, b) and (b, a). Since these two diagonals bisect one another perpendicularly, it follows that (b, a) is the reflection of (a, b) in the line $y = x$.

If this is done for every point on the graph of f it will result in the the graph of f^{-1}.

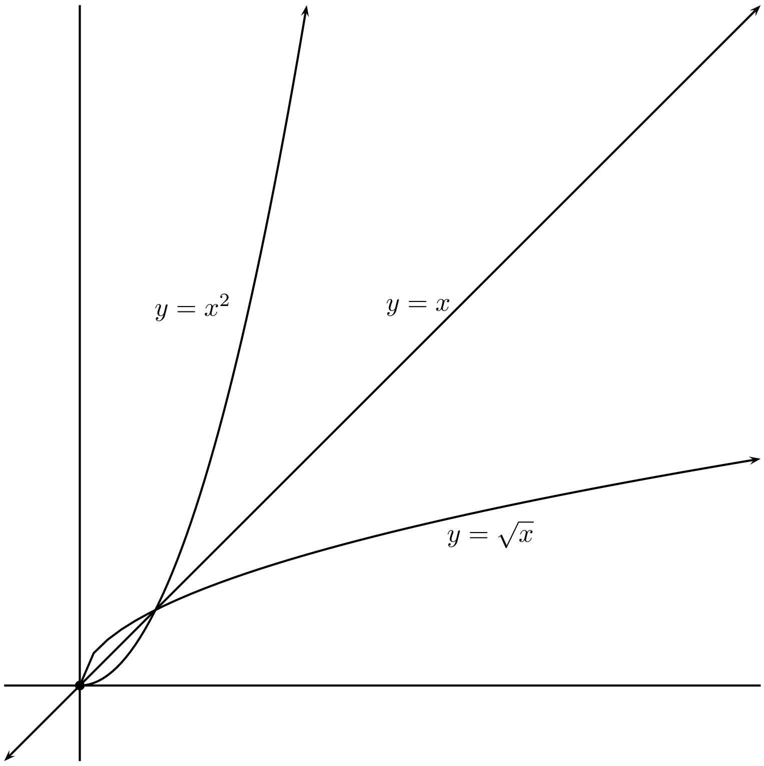
□

(iv) If f is differentiable at $a \in \mathbb{R}$ then f^{-1} will be differentiable at $f(a)$ provided that $f'(a) \neq 0$.

Example 0.9

The function $f(x) = x^2, x \in [0, \infty)$ is injective and $f^{-1}(x) = \sqrt{x}, x \in [0, \infty)$.

The following shows the graphs of x^2 and $\sqrt{x}$ along with the line $y = x$ all on the same axes:



Recall that we can use the Principle of Induction to prove that $\dfrac{d}{dx}x^n = nx^{n-1}$ for integers $n \geq 1$.
We can use this together with the Chain Rule to determine $\dfrac{d}{dx}x^{\frac{1}{n}}$:

$$\left(x^{\frac{1}{n}}\right)^n = x \Rightarrow \frac{d}{dx}\left(x^{\frac{1}{n}}\right)^n = 1.$$

Using the Chain Rule we get

$$\frac{d}{dx}\left(x^{\frac{1}{n}}\right)^n = n\left(x^{\frac{1}{n}}\right)^{n-1}\frac{d}{dx}x^{\frac{1}{n}}.$$

and so

$$n\left(x^{\frac{1}{n}}\right)^{n-1}\frac{d}{dx}x^{\frac{1}{n}} = 1 \Rightarrow \frac{d}{dx}x^{\frac{1}{n}} = \frac{1}{n\left(x^{\frac{1}{n}}\right)^{n-1}} \text{ if } x \neq 0.$$

That is,

$$\frac{d}{dx}x^{\frac{1}{n}} = \frac{1}{n}x^{\frac{1}{n}} - 1 \text{ if } x \neq 0.$$

Note that $x^{\frac{1}{n}}$ is not differentiable when $x = 0$. This is because the tangent line to the curve $y = x^{\frac{1}{n}}$ at the point $(0, 0)$ is a vertical line which therefore has no defined slope.

0.1.1.1 A function differentiable once but not twice at a point. Let $f(x) = x^{\frac{4}{3}}$. Then

$$f'(x) = \frac{4}{3}x^{\frac{1}{3}}$$

and

$$f''(x) = \frac{4}{9}x^{-\frac{2}{3}} = \frac{4}{9}\frac{1}{x^{\frac{2}{3}}}.$$

Note that $f'(0) = 0$ but $f''(0)$ does not exist. That is, this function is differentiable once but not twice at 0.